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A primer on vectors

What you should have learned in high school, but possibly forgot.

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1. Introduction

By now, you have been exposed to vectors in high school; however, some of you may have forgotten what was taught. The Ontario High School Curriculum clearly lays out what is to be taught regarding vectors and their related operations. This primer is meant to give all students the opportunity to catch up on what should have been taught in high school. We will begin by looking at two-dimensional vectors and then extend this to three dimensions. In your MATH 215 course, you will see how you can extend this to *n*-dimensional vectors and even infinite dimensional vectors; however, this primer will be sufficient for your first-year courses.

2. Two-dimensional vectors

We will begin with a definition and representation of a 2-dimensional vector and then look at some properties and operations on 2-dimensional vectors. Two-dimensional vectors are easier to represent and visualize than 3-dimensional vectors, which will follow.

2.1 Definition

A two-dimensional vector is a pair of two real numbers, for example:

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5.2 \\ 13.7 \end{pmatrix} \text{ and } \begin{pmatrix} -0.5236 \\ 1.79 \end{pmatrix}.$$

A vector can be used to describe the coordinates of a point in a plane relative to a point called the *origin* (where $x = y = 0$). For example, the first component, or coordinate, can represent the distance in the x direction, and the second is the offset in the y direction. Therefore, the vector $\begin{pmatrix} 3.73 \\ 1.85 \end{pmatrix}$ may be displayed graphically as any one of the three shown in Figure 1. The arrowhead is often referred to as the *head* of the vector while the other end, the *base*, is always at the origin. While it may be visually appealing to think of vectors as arrows, it is better to think of them as points or coordinates on the plane relative to the origin.

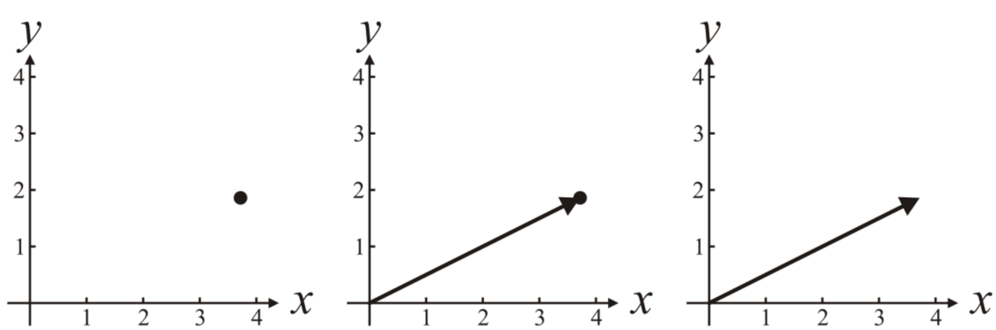


Figure 1. Representing the vector $\begin{pmatrix} 3.73 \\ 1.85 \end{pmatrix}$.

A variable understood to be a vector can be expressed as either a boldface Roman letter, an underlined letter, or a letter with an arrow above it:

$$\mathbf{v} \quad \underline{v} \quad \vec{v}$$

Textbooks tend to use the boldface variation while instructors will use one of the other two on the blackboard to differentiate vectors from variables representing, for example, real or complex numbers.

In two dimensions, given a vector such as $\mathbf{v} = \begin{pmatrix} 3.73 \\ 1.85 \end{pmatrix}$, we will often refer to the first component as the *x*-component and say that $v_1 = 3.73$; and the second as the *y*-component and say that $v_2 = 1.85$. In general, if we have a vector, say **u**, we will write its components as u_1 and u_2, respectively; that is, $\mathbf{u} \stackrel{\text{def}}{=} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$; that is, a 2-dimensional vector **u** is by definition one that has u_1 as the first component and u_2 as the second.

Textbooks tend to use italicized letters to represent the components of a vector.

If you have labeled vectors such as $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, the components are be referred to by adding a comma in the subscript. Therefore, $\mathbf{u}_1$ has components $u_{1,1}$ and $u_{1,2}$, while $\mathbf{u}_k \stackrel{\text{def}}{=} \begin{pmatrix} u_{k,1} \\ u_{k,2} \end{pmatrix}$.

Note: You may be asking, what is the difference between a coordinate and a vector? After all, are not points on the *xy* plane coordinates? What makes a coordinate different from a vector?

Essentially, a coordinate is an absolute location relative to some origin, so while one can discuss the distance between to coordinate points, it makes no sense to, for example, add two coordinates. Indeed, all of Euclidean geometry is generated without reference to concepts such as adding to coordinates, or any other operations so commonly used with vectors. A vector is a relative offset from an origin, and this relative offset has both magnitude and direction.

2.2 The length or *norm* of a vector

The length or *norm* of a vector is the length of the line from the origin in the *xy*-plane. For example, if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, we will represent the norm by $\|\mathbf{u}\|$ and

$$\|\mathbf{u}\| = \sqrt{a^2 + b^2}.$$

If we don't know the components, we would simply write $\|\mathbf{v}\| = \sqrt{v_1^{\ 2} + v_2^{\ 2}}$.

For example, if $\mathbf{u} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, then $\|\mathbf{u}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$. In general, however, norms don't always work out so easily: if $\mathbf{u} = \begin{pmatrix} -5.23 \\ 2.98 \end{pmatrix}$, then $\|\mathbf{u}\| = \sqrt{(-5.23)^2 + 2.98^2} = \sqrt{36.2333} \approx 6.01941$.

Note: there are other ways of measuring the norm of a vector. Calculating the square root of the sum of squares can be called the "Euclidean norm" and the "2-norm", and both are common. We may also denote this norm using a subscripted two, as in $\|\mathbf{u}\|_2 = \sqrt{u_1^{\ 2} + u_2^{\ 2}}$. The subscripted two simply identifies that we mean this definition of norm. Later, you may square the norm, in which case, you will see the notation $\|\mathbf{u}\|_2^{\ 2} = u_1^{\ 2} + u_2^{\ 2}$. To reinforce this, we will use the subscript-2 notation for the rest of this primer.

Note: we will always refer to the *length* of a vector as the *norm* of the vector. This is because later, we will be looking at objects that behave just like the vectors we are looking at now, but where calling such a measure the "length" would not make any sense.

2.3 Unit vectors

If the length of a vector is 1, it is said to be a *unit vector*. Four obvious unit vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ -1 \end{pmatrix};$$

however, any vector whose head lies on a circle around the origin of length 1, shown in Figure 2, is a unit vector.

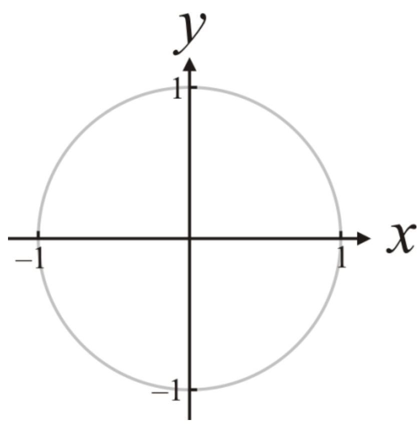


Figure 2. The unit circle around the origin.

Examples of other unit vectors include:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \text{ and } \begin{pmatrix} 0.5324 \\ 0.8465 \end{pmatrix}.$$

Now, the last one is not *exactly* a unit vector because $0.5324^2 + 0.8465^2 = 1.00001201$ and thus $\sqrt{0.5324^2 + 0.8465^2} \approx 1.000006005$, which is not exactly equal to 1, but it is very close. In engineering, we will often be using approximations.

Two very important unit vectors are given special names,

$$\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and together these are said to be the *standard basis* of the plane. The *hat* sitting on top of the variables indicates that the vectors are meant to be of unit length. These are shown in Figure 3.

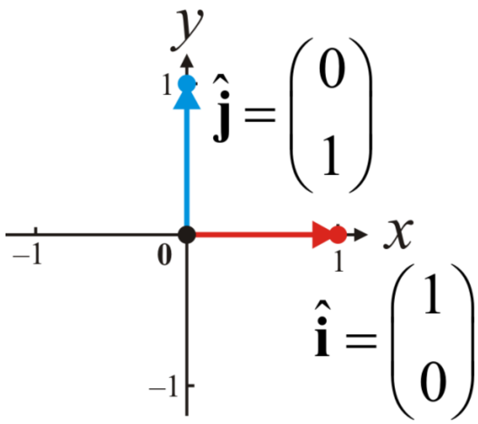


Figure 3. The two forming the *standard basis*.

Note that $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has $\|\mathbf{u}\|_2 = \sqrt{2}$, and so is not a unit vector.

2.4 The zero vector

One particularly special vector is the zero vector: a vector where both components are zero. This is often denoted using an appropriately accented zero:

$$\mathbf{0} = \underline{0} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2.5 Vector equality

Two vectors are equal if both their components are equal, so $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ are considered to be different vectors even though the first component of each are the same.

2.6 Vector addition

We add two vectors by adding their respective components. Therefore, we would say that

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+4 \\ -5+2 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$$

Graphically, if $\mathbf{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, we could show such vector addition as shown in Figure 4.

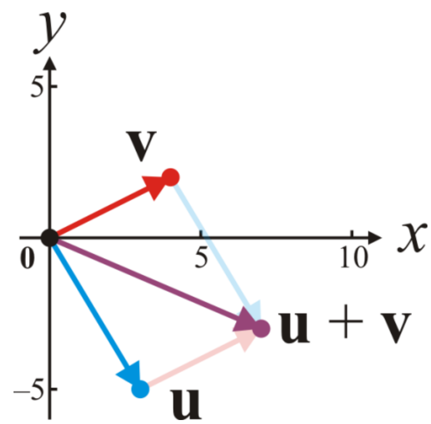


Figure 4. Adding the vectors **u** and **v**.

From the image, you may see that vector addition is commutative, that is, **u** + **v** = **v** + **u**.

Note that for each vector **v**, there is a unique vector –**v** such that **v** + (–**v**) = **0**; for example,

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is similar to each real number x being associated with its *additive inverse* where $x + (-x) = 0$; for example, 3.2 + (–3.2) = 0. (Incidentally, the *multiplicative inverse* of a non-zero value of x is $1/x$.)

Some properties of vector addition are listed here:

Property	Expression	Comment
Commutativity	**u** + **v** = **v** + **u**	
Associativity	(**u** + **v**) + **w** = **v** + (**u** + **w**)	It doesn't matter in which order you add vectors.
Identity element	**u** + **0** = **v** for all **v**	Adding the zero vector to any vector doesn't change it

Here is one place we can see how vectors differ from coordinates: **u** + **v** says, what is the offset from the origin if you first take the offset **u**, and then, from that point, offset **v** from that point, you get **u** + **v**. Consider this example: take seven steps west and three steps north, and follow this by taking two steps east and another six steps north. What is the final offset from the origin?

The individual offsets are $\mathbf{u} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, respectively; the cumulative offset is $\mathbf{u} + \mathbf{v} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$.

2.7 Scalar multiplication

We've seen previously that a vector can represent an arrow in the *xy*-plane. One thing we may want to do is either stretch or shrink a vector. For example, we may want to double the length of a vector, halve the length, or stretch it in the opposite direction. In general, if we multiply a vector by a real number, we multiply each component by that real number, so if $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$a\mathbf{u} = a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix}.$$

For example, if $\mathbf{v} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, then $2\mathbf{v} = \begin{pmatrix} 16 \\ 8 \end{pmatrix}$, $0.5\mathbf{v} = \frac{1}{2}\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, and $(-1)\mathbf{v} = -\mathbf{v} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$. These three scalar multiples are shown in Figure 5.

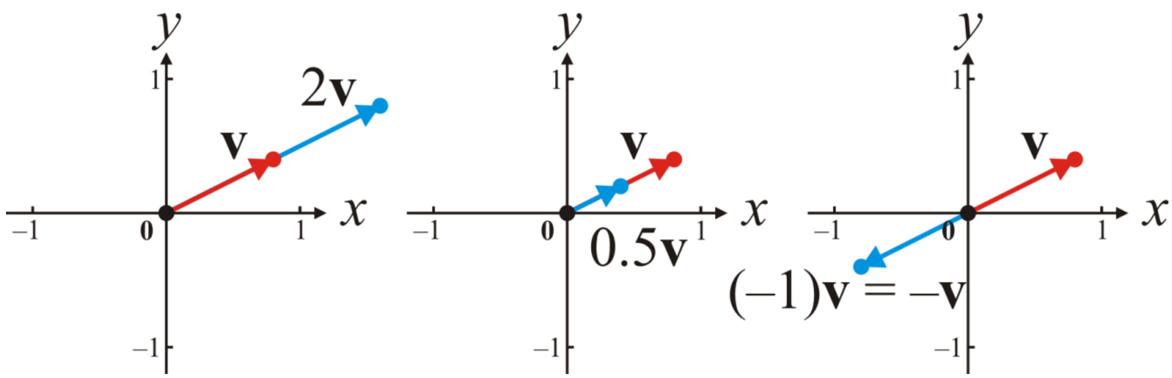


Figure 5. Three scalar multiples of a vector.

Note that $\|a\mathbf{u}\|_2 = |a|\|\mathbf{u}\|_2$; that is, if we stretch a vector by a scalar *a*, then the length of the stretched vector $a\mathbf{u}$ is $|a|$ times $\|\mathbf{u}\|_2$. Thus, if $\|\mathbf{u}\|_2 = 4.57$, then the 2-norm of –1.5**u** is 6.855; although, –1.5**u**, is *pointed* in the opposite direction of **u**.

One scalar multiplication is so common, it is given a special notation:

$$(-1)\mathbf{u} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

is usually denoted simply as –**u**. Note that –**u** is the *additive inverse* of **u**, in the sense that **u** + (–**u**) = **0**.

Some properties of scalar multiplication include:

Property	Expression	Comment
Compatibility	*a*(*b***u**) = (*ab*)**u**	The operations of scalar multiplication and the multiplication of real numbers is compatible.
Identity element	1**u** = **u**	Multiplying by 1 doesn't change a vector.

You may also deduce that 0**u** = **0** for all vectors **u**.

The operations of scalar multiplication also interact with the operations of vector addition:

Property	Expression	Comment
Distributivity	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$	It doesn't matter whether you add vectors first and then multiply by a scalar, or multiply each by a scalar and then add.
Distributivity	$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$	It doesn't matter whether you add to scalars and then apply scalar multiplication, or if scalar multiplication is applied first and the two resulting vectors are added.

One common approach is to write the vector $\begin{pmatrix} a \\ b \end{pmatrix} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. Consequently, if $\mathbf{u} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ and $\mathbf{v} = c\hat{\mathbf{i}} + d\hat{\mathbf{j}}$, then $\mathbf{u} + \mathbf{v} = (a + c)\hat{\mathbf{i}} + (b + d)\hat{\mathbf{j}}$ and $s\mathbf{u} = (sa)\hat{\mathbf{i}} + (sb)\hat{\mathbf{j}}$.

2.8 The distance between vectors

The *distance* between two vectors **u** and **v** is the length of the line segment that connects the two vectors. Note that **u** – **v** is the vector that can be added to **v** to get **u**, and therefore the distance between **u** and **v** may be calculated by $\|\mathbf{u} - \mathbf{v}\|_2$. Note also that **u** – **v** = –(**v** – **u**), so $\|\mathbf{u} - \mathbf{v}\|_2 = \|\mathbf{v} - \mathbf{u}\|_2$.

2.9 Normalizing a vector

For every vector **v** other than the zero-vector **0**, there is a unit vector which points in the same direction as **v**. This vector is the *normalized* vector of **v**. The normalization of **v** is sometimes represented as either $\hat{\mathbf{v}}$ or $\hat{\mathrm{v}}$, and we call it *vee hat*. We can find the normalization of **v** by multiplying a vector **v** by one over the norm of **v**, and this can be written as:

$$\hat{\mathbf{v}} = \hat{\mathrm{v}} = \frac{1}{\|\mathbf{v}\|_2}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}.$$

Notice that this is okay: the norm $\|\mathbf{v}\|_2$ of a vector is a real number, or a scalar. Thus, $\frac{1}{\|\mathbf{v}\|_2}$ is also a scalar, so $\frac{1}{\|\mathbf{v}\|_2}\mathbf{v}$ simply multiplies each component of **v** by the scalar $\frac{1}{\|\mathbf{v}\|_2}$.

For example, if $\mathbf{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, then

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|_2}\mathbf{u} = \frac{1}{5}\mathbf{u} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}.$$

Similarly, if $\mathbf{v} = \begin{pmatrix} -4.065 \\ -2.139 \end{pmatrix}$, then $\|\mathbf{v}\|_2 = \sqrt{(-4.065)^2 + (-2.139)^2} \approx 4.593$ and therefore

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|_2}\mathbf{v} \approx \frac{1}{4.593}\mathbf{v} = \begin{pmatrix} \frac{-4.065}{4.593} \\ \frac{-2.139}{4.593} \end{pmatrix} \approx \begin{pmatrix} -0.885 \\ -0.468 \end{pmatrix}.$$

Notice that $\sqrt{21.099546} \approx 4.593$, so we only say that it is approximately equal to, and even $\frac{-4.065}{4.593} \approx -0.885$, so again we use "$\approx$" instead of "=". We will see more about numerical approximations and numerical methods in second year.

2.10 Orthogonal vectors

Notice that if we draw two vectors in the plane, there is always one angle that is 180 degrees (π radians) or less, as is shown in Figure 6. Angles are often represented by the Greek letter "theta" or θ.

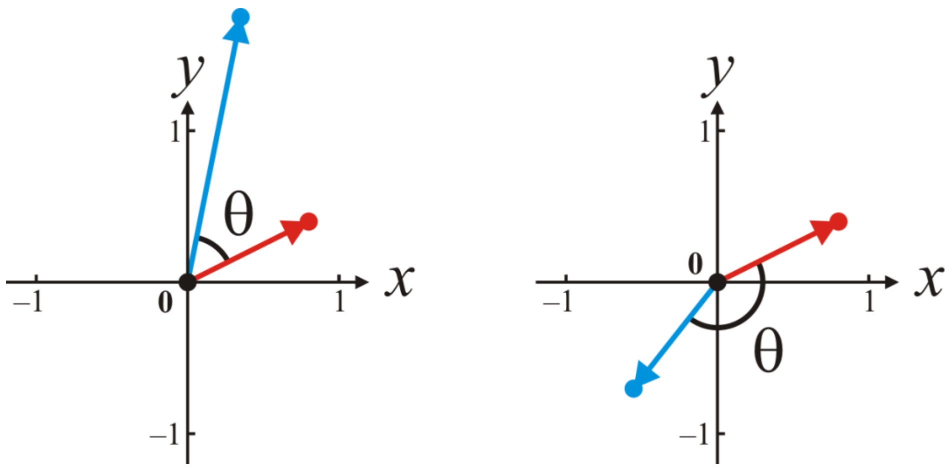


Figure 6. The angles between two vectors.

Two vectors are said to be *at right angles* to each other, or *orthogonal* to each other if the angle between them is 90 degrees or $\frac{\pi}{2}$ radians. Two such examples are shown in Figure 7.

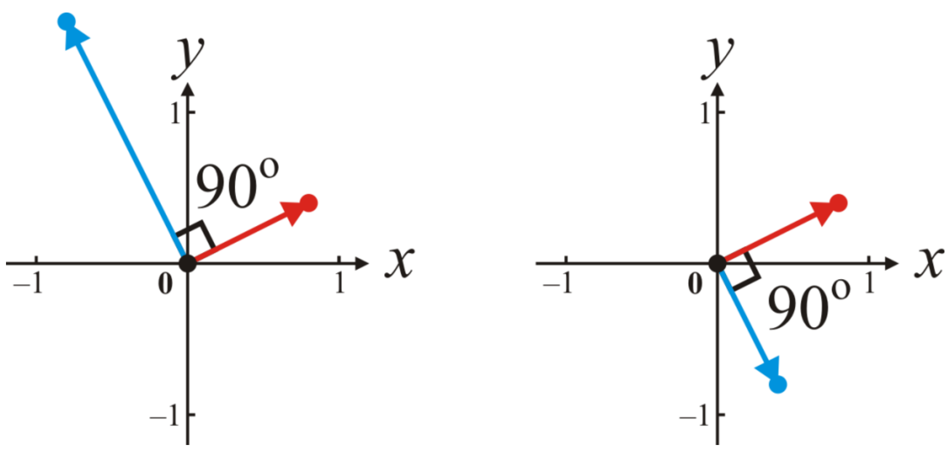


Figure 7. Two pairs of vectors that are *orthogonal* to each other.

Vectors that are at right angles to each other are very important, and we will see a means of detecting if vectors are orthogonal.

Note: the expression "**u** is orthogonal to **v**" is equivalent to the expression "**u** and **v** are orthogonal".

Some observations:

1. Given a vector $\mathbf{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}$, both $\begin{pmatrix}-u_2\\u_1\end{pmatrix}$ and $\begin{pmatrix}u_2\\-u_1\end{pmatrix}$ are orthogonal to **u**. Consider, for example, the purple vector in Figure 8, and the two vectors orthogonal to it.

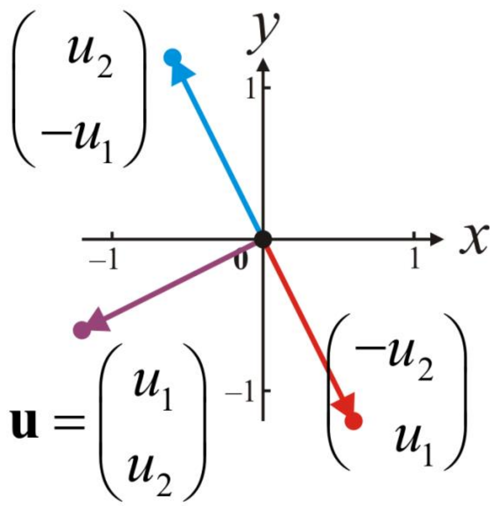


Figure 8. The vector **u** and two vectors orthogonal to it.

2. If **u** is orthogonal to **v** then any scalar multiple of **u** is orthogonal to any scalar multiple of **v**; that is, $a\mathbf{u}$ is orthogonal to $b\mathbf{v}$ for any real values a and b.
3. The zero vector **0** is orthogonal to all vectors.

2.11 Dot product or inner product

You might have already noticed that we can multiply a vector by a real number (a *scalar*), but does it make any sense to multiply a vector by a vector? For example, if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$, should we define $\mathbf{uv} = \begin{pmatrix} ac \\ bd \end{pmatrix}$? For some engineering applications, such a definition is useful (and call it *piecewise multiplication*, as we are multiplying each *piece* or *component* of the two vectors), but for linear algebra, it doesn't make much sense, so we will in general not define such an operation.

Instead, there is a far more important operation which you will use throughout your undergraduate career and throughout your professional life (even if you don't know it!), the *dot product* or *inner product*:

If $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$, we define

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = ac + bd \ .$$

That is, we multiply the components, and then add these products.

Some properties of the inner product:

Property	Expression	Comment
Commutivity	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	This is slightly different if we have complex numbers instead of real numbers as scalar values.
Linearity	$(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v})$	Linearity is a property you will see often in linear algebra and in almost all future engineering courses.
Positive definiteness	$\mathbf{0} \cdot \mathbf{0} = 0$ and $\mathbf{u} \cdot \mathbf{u} > 0$ for all other vectors $\mathbf{u}$	

2.12 The relationship between the 2-norm and the inner product

Notice that if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, then

$$\|\mathbf{u}\|_2 = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{a^2 + b^2}$$

and

$$\|\mathbf{u}\|_2^{\,2} = \mathbf{u} \cdot \mathbf{u} = \langle \mathbf{u}, \mathbf{u} \rangle = a^2 + b^2 .$$

More generally, if θ is the angle between the two vectors **u** and **v**, then

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \cos(\theta) .$$

To visualize this, consider the diagram in Figure 9 containing vectors $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$.

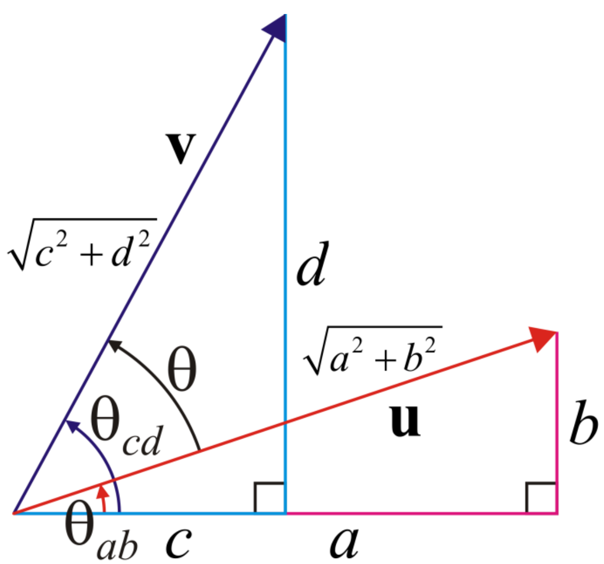


Figure 9. Two vectors, the corresponding right-angled triangles, and the various angles and lengths.

From trigonometry, you know that sine is opposite over the hypotenuse and cosine is the adjacent over the hypotenuse. Additionally, we know that $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, cosine is even (that is, cos(–*x*) = cos(*x*)), and sine is odd (that is, sin(–*x*) = –sin(*x*)). Thus, we may find that

$$\begin{aligned} \cos(\theta) &= \cos(\theta_{cd} - \theta_{ab}) \\ &= \cos(\theta_{ab})\cos(\theta_{cd}) + \sin(\theta_{ab})\sin(\theta_{cd}) \\ &= \frac{b}{\sqrt{a^2+b^2}} \frac{d}{\sqrt{c^2+d^2}} + \frac{a}{\sqrt{a^2+b^2}} \frac{c}{\sqrt{c^2+d^2}} \end{aligned}$$

Now, multiply both sides by the common denominator:

$$\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos(\theta)=bd+ac$$

and we quickly see that this is identical to

$$\|\mathbf{u}\|_2\|\mathbf{v}\|_2\cos(\theta)=\mathbf{u}\cdot\mathbf{v}.$$

2.13 The relationship between orthogonal vectors and the inner product

Now we come to one very important observations: two vectors are orthogonal to each other (that is, they are at 90 degrees or at right angles to each other) if and only if the inner product is zero. Some examples:

1. If $\mathbf{u}=\begin{pmatrix}3\\5\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}-2\\1\end{pmatrix}$, then $\mathbf{u}\cdot\mathbf{v}=3\cdot(-2)+5\cdot 1=-1$, so these vectors are not orthogonal.
2. If $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}4\\-2\end{pmatrix}$, then $\mathbf{u}\cdot\mathbf{v}=1\cdot 4+2\cdot(-2)=0$, so these vectors are orthogonal.
3. If $\mathbf{u}=\begin{pmatrix}0.5323\\0.9254\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0.8173\\-0.4701\end{pmatrix}$, then $\mathbf{u}\cdot\mathbf{v}=0.00001825$, so while these vectors are vectors not orthogonal, one may argue that they are *close to orthogonal*.

2.14 The line of all orthogonal vectors

Given a vector $\mathbf{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, any other vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ that is orthogonal to it must satisfy

$$3x - 5y = 0.$$

From high school, you will recall that this defines a line in the plane, and in this case, we can write that line as $y = \frac{3}{5}x$. Thus, if $x = 3.27$, it follows that $\mathbf{v} = \begin{pmatrix} 3.27 \\ 1.962 \end{pmatrix}$ must be also orthogonal to $\mathbf{u}$. This vector $\mathbf{u}$ and the corresponding line are shown in Figure 10.

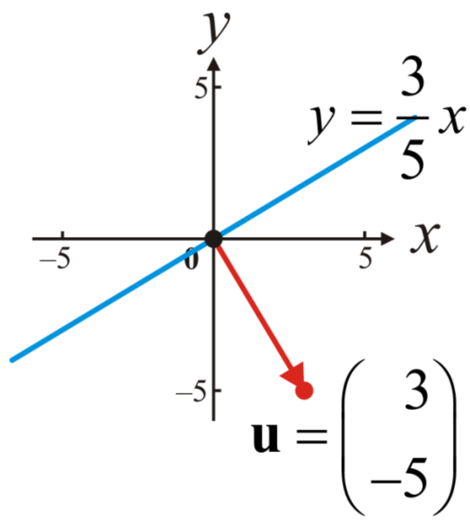


Figure 10. A vector and the line of all vectors orthogonal to it.

Note that this line always passes through the origin. Note also that in the special case where the second component is zero, for example, $\mathbf{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, the line of all orthogonal vectors falls on the line $x = 0$.

2.15 Summary of two-dimensional vectors

This section defined and looked at various operations on two-dimensional vectors. We defined the 2-norm of a vector from which we introduced unit vectors. Next, vector addition and scalar multiplication were defined followed by the normalization of non-zero vectors and the concept of the angle between vectors and orthogonality. This was concluded by defining the inner product and relating the inner product to the previous concepts. Next, we will consider three-dimensional vectors. All the operations on three-dimensional vectors are analogous to operations on two-dimensional vectors, only we will add one more: the cross product.

3. Three-dimensional vectors

We will now look at vectors that represent points in space.

3.1 Definition

A three-dimensional vector is a sequence of three real numbers, for example:

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3.2 \\ -4.5 \\ -92.4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The three components of a vector **v** may be referred to as v_1, v_2, and v_3, respectively. Every three dimensional vector represents a point in what we call *three-space* relative to an origin. In the plane (also called *two-space*), it is usually to represent positive values of x to the right, and positive values of y up. In three dimensions, positive values of x are represented as pointing out of the plane, positive values of y to the right, and positive values in z up, as shown in Figure 11.

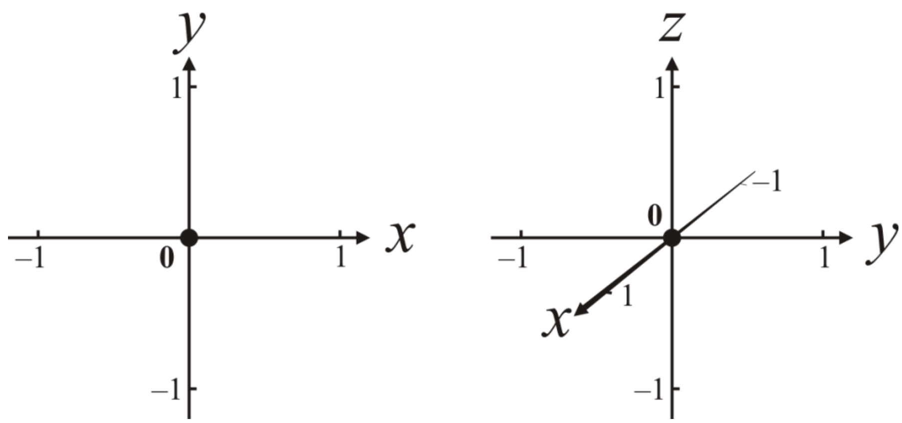


Figure 11. Representations of 2- and 3-space.

To draw vectors in three-space on a plane, we usually resort to using *guidelines* to suggest the depth of the vector. For example, the vectors $\mathbf{u} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix}$ may be shown as in Figure 12.

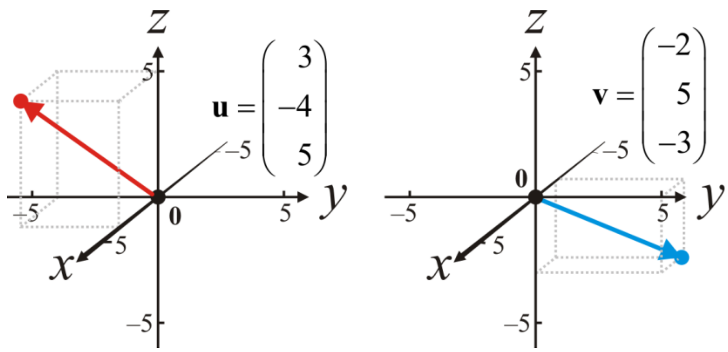


Figure 12. Giving depth to a vector in 3-space.

3.2 The 2-norm of a vector

The 2-norm of a vector in three space is the physical length of the vector:

$$\|\mathbf{u}\|_2 = \sqrt{u_1^{\,2} + u_2^{\,2} + u_3^{\,2}} \ .$$

If there is no ambiguity, the 2-norm may be simply written as $\|\mathbf{u}\|$. Note that the "2" refers to squaring the components, summing them, and then taking the square root. The "2" is not meant to suggest the dimension of the vector.

3.3 Unit vectors

If the length of a vector is 1, it is said to be a *unit vector*. Six obvious unit vectors are

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}0\\0\\-1\end{pmatrix};$$

however, any vector whose head lies on a sphere around the origin of length 1, shown in Figure 13, is a unit vector.

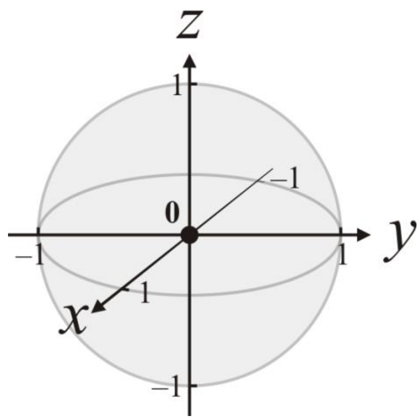


Figure 13. The unit sphere around the origin in 3-space.

Examples of other unit vectors include:

$$\begin{pmatrix}\frac{2}{7}\\-\frac{6}{7}\\-\frac{3}{7}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}, \text{ and } \begin{pmatrix}0.5324\\-0.3256\\0.7813\end{pmatrix}.$$

Again, the last one is only close to a unit vector as $0.5324^2 + 0.3256^2 + 0.7813^2 = 1.00000130$ and $\sqrt{0.5324^2 + 0.3256^2 + 0.7813^2} \approx 1.00000064$.

The standard basis for 3-space is:

$$\hat{\mathbf{i}} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \hat{\mathbf{j}} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \hat{\mathbf{k}} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

and together these are said to be the *standard basis* of the plane. These are shown in Figure 14.

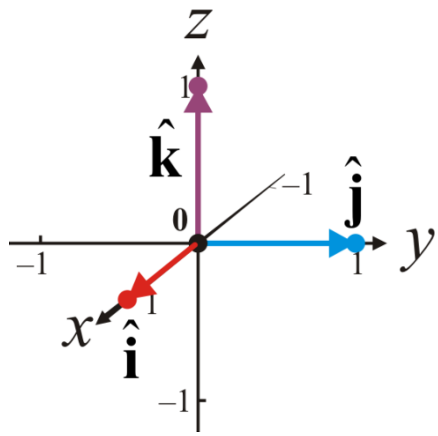


Figure 14. The two forming the *standard basis*.

3.4 The zero vector

The zero vector is as one would expect in three dimensions:

$$\mathbf{0} = \underline{0} = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3.5 Vector equality

Two 3-dimensional vectors are equal if all three components are equal. That is $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

3.6 Vector addition

We add two vectors by adding their respective components. Therefore, we would say that

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}.$$

3.7 Scalar multiplication

As in two dimensions, scalar multiplication either stretches or shrinks a vector, $a\mathbf{u} = \begin{pmatrix} a u_1 \\ a u_2 \\ a u_3 \end{pmatrix}$ and the additive inverse of $\mathbf{u}$ is $-\mathbf{u} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix}$. As before, $\|a\mathbf{u}\|_2 = |a|\|\mathbf{u}\|_2$.

As in two dimensions, one common approach is to write the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$.

Consequently, if $\mathbf{u} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{v} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, then $\mathbf{u} + \mathbf{v} = (a+d)\hat{\mathbf{i}} + (b+e)\hat{\mathbf{j}} + (c+f)\hat{\mathbf{k}}$ and $s\mathbf{u} = (sa)\hat{\mathbf{i}} + (sb)\hat{\mathbf{j}} + (sc)\hat{\mathbf{k}}$.

3.8 The distance between vectors

As in two dimension, the *distance* between two vectors **u** and **v** is $\|\mathbf{u} - \mathbf{v}\|_2 = \|\mathbf{v} - \mathbf{u}\|_2$.

3.9 Normalizing a vector

As in two dimensions, we can normalize vectors by dividing by the 2-norm:

$$\hat{\mathbf{v}} = \hat{\mathrm{v}} = \frac{1}{\|\mathbf{v}\|_2}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}.$$

For example, if $\mathbf{u} = \begin{pmatrix} 4 \\ -8 \\ -1 \end{pmatrix}$, then $\|\mathbf{u}\|_2 = \sqrt{4^2 + (-8)^2 + (-1)^2} = \sqrt{81} = 9$ and

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|_2}\mathbf{u} = \frac{1}{9}\mathbf{u} = \begin{pmatrix} \frac{4}{9} \\ -\frac{8}{9} \\ -\frac{1}{9} \end{pmatrix}.$$

3.10 Orthogonal vectors

In general, two vectors that are not scalar multiples of each other define a plane in three space, and in this plane, we can find an angle between two vectors. Like in two dimensions, if the angle is a right angle, the vectors are said to be orthogonal. There are many more vectors that are orthogonal to a given vector in three space: for example, any vector of the form $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$ is orthogonal to the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

3.11 Inner product

The inner product of two 3-dimensional vectors **u** and **v** is defined as

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Without proof, as it is much more difficult to formulate than that for two dimensions, the inner product is still equal to $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \cos(\theta)$, where θ is the angle between the two vectors in 3-space.

3.12 The relationship between the 2-norm and the inner product

As in two dimensions,

$$\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\langle \mathbf{u},\mathbf{u}\rangle} = \sqrt{u_1^{\,2} + u_2^{\,2} + u_3^{\,2}}$$

and

$$\|\mathbf{u}\|_2^{\,2} = \mathbf{u}\cdot\mathbf{u} = \langle \mathbf{u},\mathbf{u}\rangle = u_1^{\,2} + u_2^{\,2} + u_3^{\,2}.$$

3.13 The relationship between orthogonal vectors and the inner product

As in two dimensions, two vectors **u** and **v** are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = \langle \mathbf{u},\mathbf{v}\rangle = 0$.

3.14 The plane of all orthogonal vectors

Given a vector $\mathbf{u} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$, any other vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that is orthogonal to it must satisfy

$$3x - 5y + 2z = 0.$$

From high school, you will recall that this defines a plane in 3-space, and in this case, we can write that plane as $z = \frac{5}{2}y - \frac{3}{2}x$. Thus, if x = 3.27 and y = 4.92, it follows that $\mathbf{v} = \begin{pmatrix} 3.27 \\ 4.92 \\ 7.395 \end{pmatrix}$ must be also orthogonal to **u**. Note that this plane always passes through the origin.

3.15 Vectors in physics

The use of vectors in physics is so ubiquitous that special conventions are used. For example, if **x** and **y** are the position of two objects in 3-space, then the gravitational force exerted by the object at **y** by the object at **x** depends on the radius vector **r** = **y** – **x** shown in Figure 15.

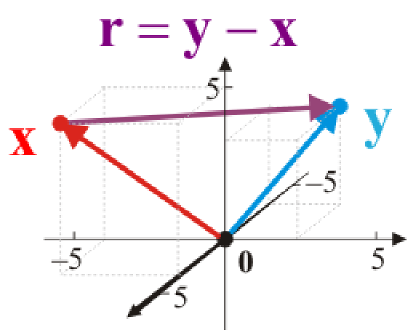


Figure 15. The radius vector r = y – x.

In this case, it is common to simply write $\|\mathbf{r}\|_2$ as r; that is, r is the scalar that represents the distance between **x** and **y**. Consequently, the gravitation force may be written as

$$F = G\frac{m_x m_y}{r^2}.$$

This force, however, has a direction, and to specify the direction, we will multiply by $\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|_2} = \frac{\mathbf{r}}{r}$, which is therefore written as

$$\mathbf{F} = G\frac{m_x m_y}{r^2}\hat{\mathbf{r}} = G\frac{m_x m_y}{r^3}\mathbf{r}.$$

3.16 The cross product

Consider any pair of vectors **u** and **v** that are not scalar multiples of each other. We would like a mechanism of defining a third vector that is orthogonal to both of these vectors. Without proof, we will claim that if we define

$$\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix},$$

then this vector is orthogonal to both **u** and **v**. You can see that it is orthogonal:

$$\begin{aligned}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\cdot\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}&=u_1\left(u_2v_3-u_3v_2\right)+u_2\left(u_3v_1-u_1v_3\right)+u_3\left(u_1v_2-u_2v_1\right)\\&=u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_2u_1v_3+u_3u_1v_2-u_3u_2v_1\\&=u_1u_2v_3-u_1u_2v_3+u_1u_3v_2-u_1u_3v_2+u_2u_3v_1-u_2u_3v_1\\&=0\end{aligned}$$

and you are welcome to convince yourself that the cross product dotted with **v** is also zero.

We will conclude with an aide-mémoire for recalling the cross product, shown in Figure 16. List the three standard basis vectors twice in the first row, and then write the components of **u** and **v** twice in each the second and third rows, respectively, and then add the product of the entries crossed in blue, and subtract the products in red:

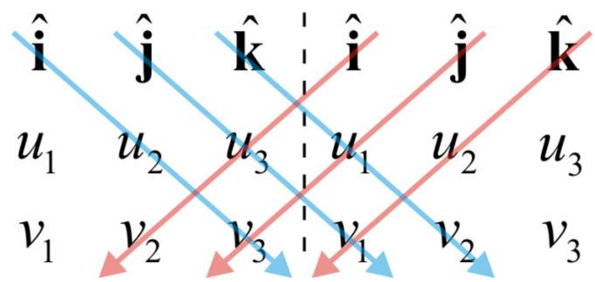
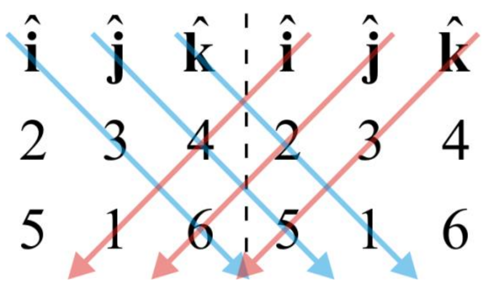


Figure 16. An aide-mémoire for the cross product.

For example, the first blue arrows crosses the three entries $\hat{\mathbf{i}}u_2v_3$, and the last red arrow crosses the entries $\hat{\mathbf{k}}u_2v_1$. Thus, we have the result

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \hat{\mathbf{i}}u_2v_3 + \hat{\mathbf{j}}u_3v_1 + \hat{\mathbf{k}}u_1v_2 - \hat{\mathbf{i}}u_3v_2 - \hat{\mathbf{j}}u_1v_3 - \hat{\mathbf{k}}u_2v_1 \\
&= \hat{\mathbf{i}}\left(u_2v_3 - u_3v_2\right) + \hat{\mathbf{j}}\left(u_3v_1 - u_1v_3\right) + \hat{\mathbf{k}}\left(u_1v_2 - u_2v_1\right) \\
&= \begin{pmatrix} u_2v_3 - u_3v_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_3v_1 - u_1v_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_1v_2 - u_2v_1 \end{pmatrix} \\
&= \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}
\end{aligned}$$

For example, in order to calculate the cross product of the vectors $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$, we create the above grid



and thus our answer is

$$\begin{aligned}
&(3\cdot 6)\hat{\mathbf{i}} + (4\cdot 5)\hat{\mathbf{j}} + (2\cdot 1)\hat{\mathbf{k}} - (4\cdot 1)\hat{\mathbf{i}} - (2\cdot 6)\hat{\mathbf{j}} - (3\cdot 5)\hat{\mathbf{k}} \\
&= (18-4)\hat{\mathbf{i}} + (20-12)\hat{\mathbf{j}} + (2-15)\hat{\mathbf{k}} \\
&= 14\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 13\hat{\mathbf{k}}
\end{aligned}$$

or

$$\begin{pmatrix} 3\cdot 6 - 4\cdot 1 \\ 4\cdot 5 - 2\cdot 6 \\ 2\cdot 1 - 3\cdot 5 \end{pmatrix} = \begin{pmatrix} 18-4 \\ 20-12 \\ 2-15 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \\ -13 \end{pmatrix}.$$

Some properties of the cross product deal with the standard basis vectors are:

$$\begin{aligned} \hat{\mathbf{i}}\times\hat{\mathbf{j}}&=\hat{\mathbf{k}} & \hat{\mathbf{j}}\times\hat{\mathbf{i}}&=-\hat{\mathbf{k}} \\ \hat{\mathbf{j}}\times\hat{\mathbf{k}}&=\hat{\mathbf{i}} & \hat{\mathbf{k}}\times\hat{\mathbf{j}}&=-\hat{\mathbf{i}} \\ \hat{\mathbf{k}}\times\hat{\mathbf{i}}&=\hat{\mathbf{j}} & \hat{\mathbf{i}}\times\hat{\mathbf{k}}&=-\hat{\mathbf{j}} \end{aligned}$$

Other properties also include:

Property	Expression	Comment
Anti-commutitivity	$\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$	Scalar multiplication is commutative: $ab = ba$.
Linearity	$(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$ $(a\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(a\mathbf{v})=a(\mathbf{u}\times\mathbf{v})$	Again, linearity is a property you will see often in linear algebra and in almost all future engineering courses.

In general, it can also be shown that

$$\|\mathbf{u}\times\mathbf{v}\|_2=\|\mathbf{u}\|_2\|\mathbf{v}\|_2\sin(\theta);$$

therefore, the angle between the vectors is 0 (they are scalar multiples of each other) if and only if the cross product is the zero vector. Note that we do not require absolute values around the sin(θ) as the angle between two vectors is always some value between 0 and π.

3.17 Right-hand rule

You might be wondering which direction the cross product of two vectors goes. The easiest way to us this is to use the *right-hand rule*. With your right hand, hold your four fingers together and make your thumb perpendicular to them. Now, point your four fingers in the same direction as **u**. Next, rotate your hand so that you can curl your fingers toward the vector **v** through the plane defined by **u** and **v** so that your thumb continues to point in the same direction. In this case, your thumb is pointing in the direction of $\mathbf{u}\times\mathbf{v}$.

You will use the right-hand rule in other circumstances: if your thumb is pointing in the direction of a current moving through a wire, any magnetic field will curl around the wire in the direction that your right hand would curl around your thumb. This is shown in Figure 17.

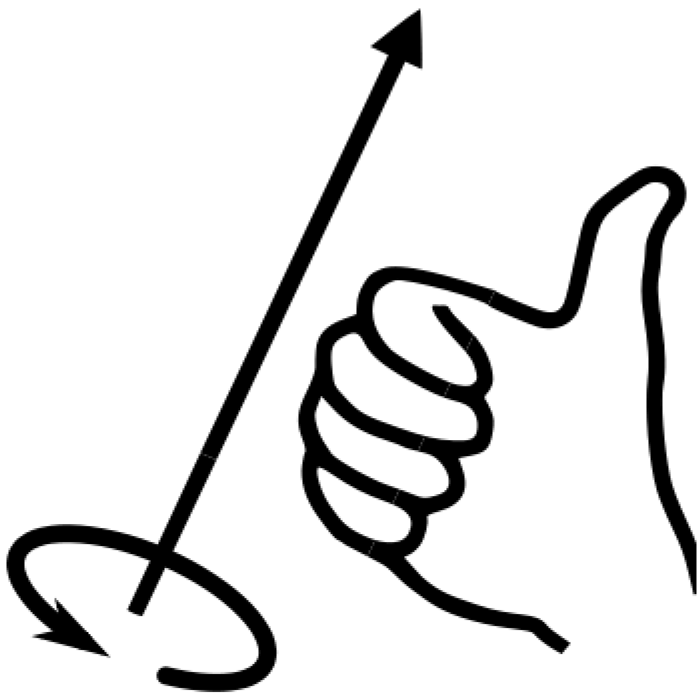


Figure 17. The right-hand rule, image by Benutzer:Schorschi2.

3.18 Summary of three-dimensional vectors

We have looked at three-dimensional vectors and seen that they are very similar to two-dimensional vectors. The only significant difference is that in three dimensions, we can define the cross product of two vectors and this will produce a third vector orthogonal to both of the first two vectors. The next three sections include a look ahead to other concepts, using Matlab for doing mathematics and vectors, a summary sheet, and questions followed by answers.

4. Looking ahead

If you have time, ask yourself what a 4-dimensional vector would look like, or more general an n-dimensional vector where n is any natural number.

Also, recall that we defined vector addition and scalar multiplication. You might want to consider two real-valued functions of a real value, f and g. Given each of these, we define $f(x)$ and $g(x)$ to represent the function evaluated at the point x. This is like a vector **v** where v_k is the k^{th} component of **v**. With two vectors, we can define **u** + **v** where the k^{th} component of this vector sum is $u_k + v_k$. Similarly, we may define the function $f + g$, and this evaluated at x is defined as the sum of both f and g evaluated at x; that is, $(f + g)(x) = f(x) + g(x)$. Similarly, given a vector **u**, we can define $a\mathbf{u}$ where the k^{th} component is au_k. Similarly, we may define af where $(af)(x) = af(x)$. There is also a zero function f_0 where $f_0(x) = 0$ for all values of x. We can define the additive inverse of a function f as $(-f)(x) = -f(x)$. Basically, this is a small hint that it appears that the same operations that work on vectors also work on functions. You will look more into this later.

5. Matlab

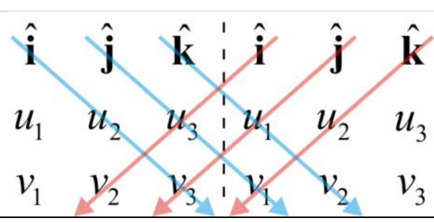
By the time you graduate, you will have learned the Matlab mathematical programming language. This section is not required for first-year students, but some of you who are more keen may consider reading this section. First, when you launch Matlab, you are confronted with an interpreter: you will type a command and Matlab will execute the command. Here, >> is the prompt, input is in black, comments in green, and output in cyan.

```
>> 3 + 7*(4 + 3) - 4.5     % Matlab can be used as a fancy calculator
  ans =
     47.5000

>> exp( -j*pi )            % e^x is represented by exp(x)
  ans =
    -1.0000 - 0.0000i

>> j^2                     % if not used as variables, i == j == sqrt( -1 )
  ans =
    -1

>> sin( 3.2 )^2 + cos( 3.2 )^2   % it is aware of trig
  ans =
       1

>> cos( pi )
  ans =
      -1

>> v = [3 5]'      % create a vector and assign it to the variable 'v'
  v =
       3
       5

>> u = [-2 3]'
  u =
      -2
       3

>> v + u           % add these two vectors
  ans =
       1
       8

>> 3.2 * u         % multiply the vector u by the scalar 3.2
  ans =
     -6.4000
      9.6000
```

```
>> norm( u - v )          % calculate the distance between u and v
  ans =
      5.3852

>> norm( u )              % calculate the 2-norm of u
  ans =
      3.6056

>> norm( -4*u )           % calculate the norm of -4*u
  ans =
     14.4222

>> 4 * norm( u )          % calculate 4 times the norm of u
  ans =
     14.4222

>> z2 = zeros( 2, 1 )     % create the 2-dimensional zero vector
  z2 =
       0
       0

>> w = [-2 3]'
  w =
      -2
       3

>> u == w                  % compare the entries componentwise
  ans =
       1
       1

>> all( u == w )           % determine if u == v (1 is true, 0 is false)
  ans =
       1

>> u = u/norm(u)           % normalize u
  u =
     -0.5547
      0.8321

>> v'*w
  ans =
       9

>> w'*v
  ans =
       9

>> u'*w
  ans =
      3.6056
```

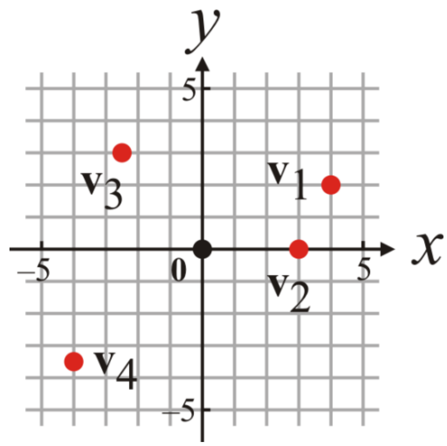
```
>> % The angle between v and w is approximately 64.653824058053306 degrees
>> % or 1.1284221038181517 radians
>> norm( v ) * norm( w ) * cos( 1.1284221038181517 )
  ans =
      9.0000

>> format long              % we need to see more precision...
>> % every result not assigned to something is assigned to the variable 'ans'
>> ans
  ans =
     9.000000000000002

>> u = [1 2 3]'             % a three-dimensional vector
  u =
       1
       2
       3

>> v = [4 3 -1]'
  v =
       4
       3
      -1

>> u'*v                     % these vectors are not orthogonal
  ans =
       7

>> cross( u, v )            % calculate the cross prouct
  ans =
     -11
      13
      -5

>> w = cross( v, u )        % (u x v) = -(v x u)
  w =
      11
     -13
       5

>> w'*v                     % u x v is orthogonal to both u and v
  ans =
       0

>> w'*u
  ans =
       0
```

6. Summary Sheet

	Two dimensions	Three dimensions
Definition	$\mathbf{u}=\vec{u}=\underline{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}$, $\mathbf{u}_k=\vec{u}_k=\underline{u}_k=\begin{pmatrix}u_{k,1}\\u_{k,2}\end{pmatrix}$	$\mathbf{u}=\vec{u}=\underline{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$, $\mathbf{u}_k=\vec{u}_k=\underline{u}_k=\begin{pmatrix}u_{k,1}\\u_{k,2}\\u_{k,3}\end{pmatrix}$
2-norm	$\lVert\mathbf{u}\rVert_2=\sqrt{u_1^2+u_2^2}$	$\lVert\mathbf{u}\rVert_2=\sqrt{u_1^2+u_2^2+u_3^2}$
Unit vectors	$\lVert\mathbf{u}\rVert_2=1$	
Standard basis	$\hat{\mathbf{i}}=\begin{pmatrix}1\\0\end{pmatrix}$, $\hat{\mathbf{j}}=\begin{pmatrix}0\\1\end{pmatrix}$	$\hat{\mathbf{i}}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\hat{\mathbf{j}}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\hat{\mathbf{k}}=\begin{pmatrix}0\\0\\1\end{pmatrix}$
The zero vector	$\mathbf{0}=\vec{0}=\underline{0}=\begin{pmatrix}0\\0\end{pmatrix}$	$\mathbf{0}=\vec{0}=\underline{0}=\begin{pmatrix}0\\0\\0\end{pmatrix}$
Vector equality	$\mathbf{u}=\mathbf{v}$ if and only if $u_1=v_1$ and $u_2=v_2$	$\mathbf{u}=\mathbf{v}$ if and only if $u_1=v_1$, $u_2=v_2$ and $u_3=v_3$
Vector addition	$\mathbf{u}+\mathbf{v}=\begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}$	$\mathbf{u}+\mathbf{v}=\begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}$
Scalar multiplication	$a\mathbf{u}=\begin{pmatrix}au_1\\au_2\end{pmatrix}$	$a\mathbf{u}=\begin{pmatrix}au_1\\au_2\\au_3\end{pmatrix}$
Distance between vectors	The distance between **u** and **v** is $\lVert\mathbf{u}-\mathbf{v}\rVert_2=\lVert\mathbf{v}-\mathbf{u}\rVert_2$	
Normalization	If $\mathbf{u}\neq\mathbf{0}$, $\hat{\mathbf{u}}=\dfrac{\mathbf{u}}{\lVert\mathbf{u}\rVert_2}$	
Inner product	$\mathbf{u}\cdot\mathbf{v}=\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1+u_2v_2$ $=\lVert\mathbf{u}\rVert_2\lVert\mathbf{v}\rVert_2\cos(\theta)$	$\mathbf{u}\cdot\mathbf{v}=\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1+u_2v_2+u_3v_3$ $=\lVert\mathbf{u}\rVert_2\lVert\mathbf{v}\rVert_2\cos(\theta)$
Orthogonality	**u** and **v** are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v}=0$	
The inner product and the 2-norm	$\lVert\mathbf{u}\rVert_2=\sqrt{\mathbf{u}\cdot\mathbf{u}}$	
Vectors orthogonal to a vector	All vectors $\begin{pmatrix}x\\y\end{pmatrix}$ orthogonal to **u** satisfy $xu_1+yu_2=0$ and form a line passing through the origin perpendicular to **u**	All vectors $\begin{pmatrix}x\\y\\z\end{pmatrix}$ orthogonal to **u** satisfy $xu_1+yu_2+zu_3=0$ and form a plane passing through the origin perpendicular to **u**
Cross product (for 3-dimensional vectors only)		$\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}$ $\lVert\mathbf{u}\times\mathbf{v}\rVert_2=\lVert\mathbf{u}\rVert_2\lVert\mathbf{v}\rVert_2\sin(\theta)$

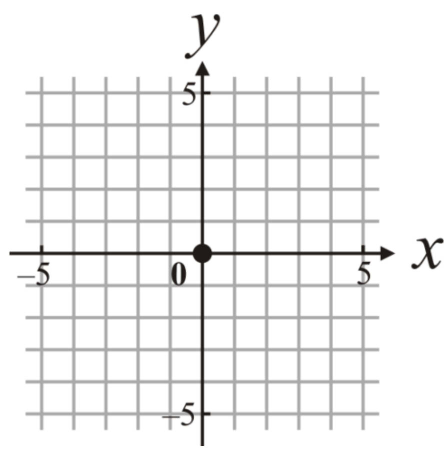
7. Questions for 2-dimensional vectors

1. Determine the components of the following five vectors.



2. Plot the following four vectors on the plane provided.

$$\mathbf{u}_1 = \begin{pmatrix} -2.8 \\ 3.4 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} 2.9 \\ -2.6 \end{pmatrix},\ \mathbf{u}_3 = \begin{pmatrix} -1.1 \\ -1.7 \end{pmatrix},\ \mathbf{u}_3 = \begin{pmatrix} 0 \\ -0.9 \end{pmatrix}.$$



3. What is the 2-norm of each of the following vectors? You may require a calculator.

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix},\ \mathbf{u}_3 = \begin{pmatrix} 8 \\ 4 \end{pmatrix},\ \mathbf{u}_4 = \begin{pmatrix} -2.1 \\ -1.3 \end{pmatrix}$$

4. If the first component of a vector **u** is $u_1 = 0.532$ and **u** is known to be a unit vector, what are the two possible values of the second component?
5. Is it correct to say that for a 2-dimensional vector, $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}$?
6. Suppose that **u** is a 2-dimensional vector. What happens if we define $\mathbf{v} = u_1\hat{\mathbf{i}}$? What can we say about the vector **v** in relation to the vector **u**?
7. Is $0\mathbf{u} = \mathbf{0}$ for all vectors **u**?

8. Consider the vector $\mathbf{u} = \begin{pmatrix} x \\ x+y \end{pmatrix}$. For what values of *x* and *y* is this vector equal to $\mathbf{v} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$?

9. Consider the vector $\mathbf{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. For what values of θ is this vector equal to $\mathbf{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$?

10. Calculate the sum $\begin{pmatrix} 3.7 \\ 4.2 \end{pmatrix} + \begin{pmatrix} -2.5 \\ -1.3 \end{pmatrix} + \begin{pmatrix} -5.2 \\ 1.3 \end{pmatrix} + \begin{pmatrix} 4.5 \\ -2.1 \end{pmatrix}$.

11. Calculate the sum $0.9\begin{pmatrix} 3.7 \\ 4.2 \end{pmatrix} + 0.1\begin{pmatrix} 4.5 \\ -2.1 \end{pmatrix}$.

12. Calculate the sum $0.5\left(\begin{pmatrix} 3.7 \\ 4.2 \end{pmatrix} + \begin{pmatrix} 4.5 \\ -2.1 \end{pmatrix}\right)$. How would you describe this vector in relation to the two vectors being summed?

13. What is the distance between the vectors $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

14. For which values of *x* is the distance between the two vectors $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} x \\ 5 \end{pmatrix}$ equal to 3?

15. Normalize the vectors $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ -15 \end{pmatrix}$.

16. Find approximations of the normalization of the vectors $\begin{pmatrix} 4.3 \\ -2.5 \end{pmatrix}$ and $\begin{pmatrix} -5.3 \\ 2.8 \end{pmatrix}$.

17. Find the inner product of the vector $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ with the vectors $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$.

18. Are the following pairs of vectors orthogonal?

$$\begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

19. For what values of *x* is the vector $\begin{pmatrix} x \\ 6 \end{pmatrix}$ orthogonal to the vector $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

20. If $\mathbf{u} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$, what is $\mathbf{u} \cdot \mathbf{u}$? What is $\|\mathbf{u}\|_2^2$?

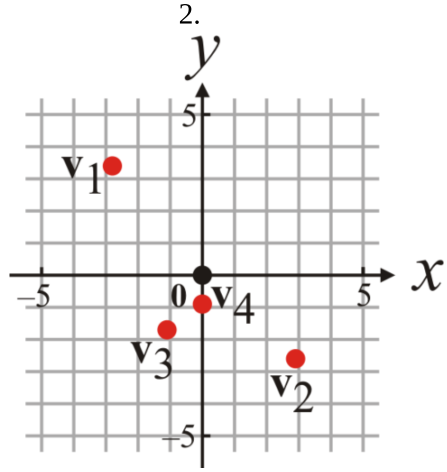
21. What is the equation of the line orthogonal to the vector $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$?

22. What is the equation of the line orthogonal to the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$?

8. Answers to questions on 2-dimensional vectors

1. $\mathbf{u}_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} -2.5 \\ 3 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} -4 \\ -3.5 \end{pmatrix}$

2.



3. 3.6056, 2.2361, 8.9443, 2.4698
4. The second component would be approximately 0.8467 or –0.8467.
5. Yes.
6. The vector $\mathbf{v} = u_1\hat{\mathbf{i}}$ is the closest vector on the *x*-axis to the vector **u**.
7. Yes.
8. $x = -1$ and $y = -2$.
9. θ can be equal to $\frac{3}{2}\pi$, but it can also any value of the form $\frac{3}{2}\pi + 2\pi n$ for any integral value of *n*.
10. $\begin{pmatrix} 0.5 \\ 2.1 \end{pmatrix}$.
11. $\begin{pmatrix} 3.78 \\ 3.57 \end{pmatrix}$.
12. $\begin{pmatrix} 4.1 \\ 1.05 \end{pmatrix}$ is the vector lying half-way between the two given vectors.
13. $\sqrt{29} \approx 5.3852$
14. $3 - 2\sqrt{2} \approx 0.1716$ and $3 + 2\sqrt{2} \approx 5.8284$.
15. $\begin{pmatrix} \frac{-5}{13} \\ \frac{12}{13} \end{pmatrix}$ and $\begin{pmatrix} \frac{8}{17} \\ \frac{-15}{17} \end{pmatrix}$.
16. $\begin{pmatrix} 0.8645 \\ -0.5026 \end{pmatrix}$ and $\begin{pmatrix} -0.8842 \\ 0.4671 \end{pmatrix}$.
17. 18, 27, 36
18. Yes, no, and yes.
19. $x = 12$

20. In both cases, 169
21. $y = 4x$
22. $x = 0$

9. Questions for 3-dimensional vectors

1. What is the 2-norm of each of the following vectors? You may require a calculator.

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix},\ \mathbf{u}_3 = \begin{pmatrix} 0.2 \\ -0.5 \\ 7.0 \end{pmatrix},\ \mathbf{u}_4 = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$$

2. If the first two components of a vector **u** are $u_1 = 0.532$ and $u_2 = 0.452$ and **u** is known to be a unit vector, what are the two possible values of the third component?
3. Is it correct to say that for a 3-dimensional vector, $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$?
4. Suppose that **u** is a three-dimensional vector. What happens if we define $\mathbf{v} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}$? What can we say about the vector **v** in relation to the vector **u**?
5. Is $0\mathbf{u} = \mathbf{0}$ for all 3-dimensional vectors **u**?
6. Consider the vector $\mathbf{u} = \begin{pmatrix} x \\ x+y \\ x+z \end{pmatrix}$. For what values of x, y and z is this vector equal to $\mathbf{v} = \begin{pmatrix} 5.2 \\ -4.1 \\ 6.5 \end{pmatrix}$?
7. Consider the vector $\mathbf{u} = \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ \sin(\phi) \end{pmatrix}$. For what values of θ is this vector equal to $\mathbf{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$?
8. Calculate the sum $\begin{pmatrix} 3.2 \\ -4.7 \\ 1.3 \end{pmatrix} + \begin{pmatrix} 5.2 \\ 1.2 \\ -0.5 \end{pmatrix} + \begin{pmatrix} -4.7 \\ 0.2 \\ 0.5 \end{pmatrix} + \begin{pmatrix} -5.2 \\ 4.7 \\ -1.5 \end{pmatrix}$.
9. Calculate the sum $0.9\begin{pmatrix} 2.3 \\ -0.5 \\ 6.9 \end{pmatrix} + 0.1\begin{pmatrix} 9.8 \\ 0.7 \\ -8.5 \end{pmatrix}$.
10. Calculate the sum $0.5\left(\begin{pmatrix} 2.3 \\ -0.5 \\ 6.9 \end{pmatrix} + \begin{pmatrix} 9.8 \\ 0.7 \\ -8.5 \end{pmatrix}\right)$. How would you describe this vector in relation to the two vectors being summed?
11. What is the distance between the vectors $\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 6 \\ -8 \end{pmatrix}$?
12. For which values of x is the distance between the two vectors $\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} x \\ 1 \\ -3 \end{pmatrix}$ equal to 3?
13. Normalize the vectors $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -7 \\ 4 \end{pmatrix}$.

14. Find approximations of the normalization of the vectors $\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2.1 \\ -1.3 \\ -1.7 \end{pmatrix}$.

15. Find the inner product of the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ with the vectors $\begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 0.3 \\ 0.7 \\ -0.5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

16. Are the following pairs of vectors orthogonal?

17. For what values of x is the vector $\begin{pmatrix} x \\ 6 \\ 2 \end{pmatrix}$ orthogonal to the vector $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$?

18. If $\mathbf{u} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$, what is $\mathbf{u} \cdot \mathbf{u}$? What is $\|\mathbf{u}\|_2^2$?

19. What is the equation of the plane orthogonal to the vector $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$?

20. What is the equation of the plane orthogonal to the vector $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$?

21. What is the cross product of the following pairs of vectors:

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

10. Answers to questions on 3-dimensional vectors

1. 3, $\sqrt{77} \approx 8.7750$, 7.0207, and 8.9443.
2. 0.7160 and –0.7160
3. Yes.
4. **v** is the closest point on the *xy*-plane to the vector **u**.
5. Yes.
6. *x* = 5.2, *y* = –9.3, and *z* = 1.3
7. Consider the vector $\mathbf{u} = \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ \sin(\phi) \end{pmatrix}$. For what values of θ is this vector equal to $\mathbf{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$?
8. Calculate the sum $\begin{pmatrix} -1.5 \\ 1.4 \\ -0.2 \end{pmatrix}$.
9. Calculate the sum $\begin{pmatrix} 3.05 \\ -0.38 \\ 5.36 \end{pmatrix}$.
10. $\begin{pmatrix} 6.05 \\ 0.10 \\ -0.80 \end{pmatrix}$ and it is the midpoint between the two vectors.
11. $\sqrt{209} \approx 14.4568$
12. $3 - \sqrt{7}$ and $3 + \sqrt{7}$
13. $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} -\frac{4}{9} \\ -\frac{7}{9} \\ -\frac{4}{9} \end{pmatrix}$.
14. $\begin{pmatrix} 0.3841 \\ 0.5121 \\ -0.7682 \end{pmatrix}$ and $\begin{pmatrix} 0.7004 \\ -0.4336 \\ -0.5670 \end{pmatrix}$.
15. –20, -2.6, 3.
16. Are the following pairs of vectors orthogonal?
17. $-\frac{26}{3}$
18. Both are 62.
19. $z = \frac{1}{3}x - \frac{2}{3}y$

20. $y = -\frac{3}{2}x$ with z taking on all values; that is, $\begin{pmatrix} x \\ -\frac{3}{2}x \\ z \end{pmatrix}$.

21. What is the cross product of the following pairs of vectors:

$$\begin{pmatrix} 4 \\ 0 \\ -12 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -16 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

11. Acknowledgments

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